Stabilizing discontinuous Galerkin schemes. Applications to Shallow-Water equations.

Sacha Cardonna

Advised by François Vilar & Fabien Marche

UNIVERSITÉ DE MONTPELLIER
Institut Montpelliérain Alexander Grothendieck

July 4^{th} 2023

Subject

Finite-Volume subcell correction
for discontinuous Galerkin schemes.
Applications to Shallow-Water equations.

Goals.

- Studying high-order discontinuous Galerkin schemes;
- Stabilizing numerical schemes using an *a priori* correction;
- Theoretical study of Shallow-Water equations.

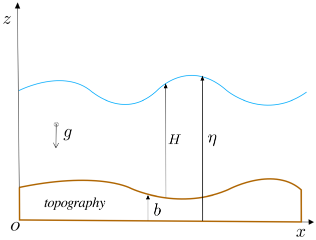
Outline of the presentation

1. Shallow-Water equations
   - Benefits of the model
   - Mathematical formulation
2. High-order discontinuous Galerkin schemes
   - Overview and advantages
   - Limits for hyperbolic and non-linear equations
3. Finite-Volume subcell correction
   - Reformulating dG as FV-like
   - *A posteriori* & *a priori* approaches
4. Stabilization of Shallow-Water equations
   - Subcell formulation and water-height positivity
   - Internship continuation and Ph.D. goals

I. Shallow-Water equations

Benefits of Shallow-Water model

(a) Grau du Roi's coastline.



(b) Shallow-Water notations.

- Used mainly in oceanography, hydrology and fluid mechanics;
- Derived from more complex Navier-Stokes equations;
- Allow efficient big-scale simulations in real time.

Pre-balanced formulation

First-order hyperbolic system with source term :

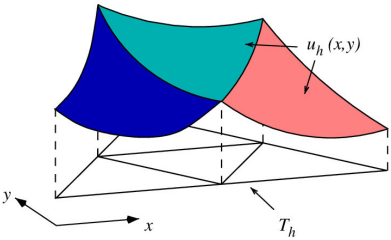
$$
\begin{aligned}
&\partial_t \boldsymbol{v} + \partial_x \boldsymbol{F}(\boldsymbol{v}, b) = \boldsymbol{B}(\boldsymbol{v}, \partial_x b) \\
\Leftrightarrow &\begin{cases} \partial_t \eta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(uq + \frac{g(\eta^2 - 2\eta b)}{2} \right) = -g\eta\partial_x b. \end{cases}
\end{aligned}
$$

- $b : \mathbb{R} \to \mathbb{R}$ is the *topography* parametrization ;
- $\boldsymbol{v} : \mathbb{R} \times \mathbb{R}_+ \to \Theta$ is the vector gathering *total elevation* η and *discharge* q, with $\Theta = \{(\eta, q) \in \mathbb{R}^2 \mid H := \eta - b \geq 0\}$;
- $\boldsymbol{F} : \Theta \times \mathbb{R} \to \mathbb{R}^2$ is the nonlinear flux funcion ;
- $\boldsymbol{B} : \Theta \times \mathbb{R} \to \mathbb{R}^2$ is the source term depending on topography.

$\rightarrow$ Theoretical derivation of Shallow-Water from incompressible Euler equations.

II. Discontinuous Galerkin schemes

Overview of dG schemes and advantages



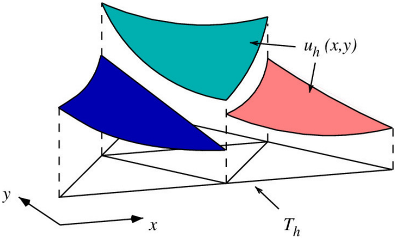


Figure: Continuous and discontinuous Galerkin methods.

- High-order schemes, well-adapted to unstructured meshes;
- Applied to various problems, including hyperbolic PDEs;
- Local formulation interesting for parallel computating.

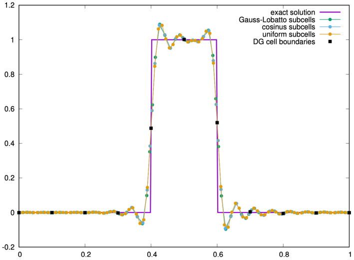
→ Development *from scratch* of a simple C++ program for solving scalar conservation laws using high-order dG (available on Git).

Limits for hyperbolic and non-linear equations

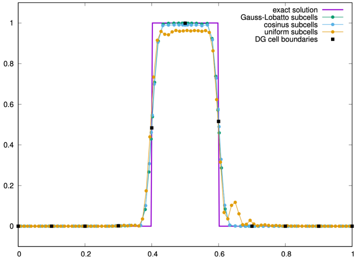
⚠ Presence of non-physical oscillations when approaching discontinuities or strong gradients, leading to :

$\rightarrow$ Non preservation of maximum principle;

$\rightarrow$ Loss of water height positivity in Shallow-Water context.



(a) Without correction.



(b) With FVS correction.

Figure: Numerical solution of $\partial_t u + c\partial_x u = 0$, with $u_0(x,t) := \mathbf{1}_{[0.4,0.6]}$.

III. Finite-Volume subcell correction

Principle

Goals. Refine the geometry by **dividing each cell into subcells**, and use *first-order* **Finite-Volume** scheme in order to handle the robustness issues.

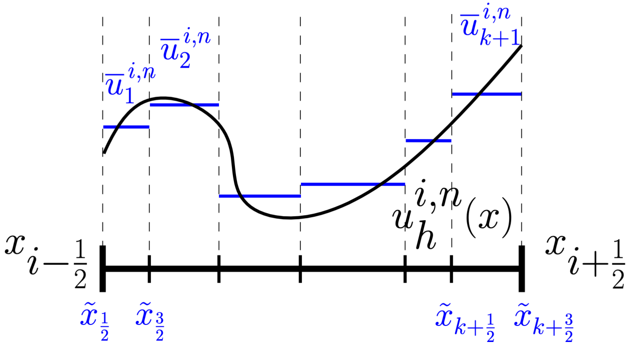


Figure: Sub-mean values on cell $\omega_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$.

Reformulating dG as FV-like on SCL

Needs. Reformulating dG schemes as *Finite-Volume like* schemes at subcell scale using sub-mean values, i.e. the formulation on cell ω_i

$$\int_{\omega_i} \varphi \partial_t u_h^{\omega_i} \,\mathrm{d}x = \int_{\omega_i} f(u_h^{\omega_i}) \partial_x \varphi \,\mathrm{d}x - \left[\mathscr{F}\varphi\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}, \qquad \forall \varphi \in \mathbb{P}^k(\omega_i),$$

gives us the local formulations on subcells $S_m^{\omega_i}$

$$\partial_t \overline{u}_m^{\omega_i} = -\frac{1}{|S_m^{\omega_i}|} \left(\widehat{F}^{\omega_i}_{m+\frac{1}{2}} - \widehat{F}^{\omega_i}_{m-\frac{1}{2}} \right), \qquad \forall m \in [\![1, k+1]\!],$$

where the $k+2$ *reconstructed fluxes* $\widehat{F}^{\omega_i}_{m+\frac{1}{2}}$ are defined by

$$\begin{aligned} \widehat{F}^{\omega_i}_{m+\frac{1}{2}} = f_h^{\omega_i}(\widetilde{x}^{\omega_i}_{m+\frac{1}{2}}) &- C^{i-\frac{1}{2}}_{m+\frac{1}{2}} \left(f_h^{\omega_i}(x_{i-\frac{1}{2}}) - \mathscr{F}_{i-\frac{1}{2}} \right) \\ &- C^{i+\frac{1}{2}}_{m+\frac{1}{2}} \left(f_h^{\omega_i}(x_{i+\frac{1}{2}}) - \mathscr{F}_{i+\frac{1}{2}} \right). \end{aligned}$$

A posteriori approach

- Introduced on Shallow-Water during Ali Haidar's PhD;
- Replacing reconstructed fluxes with 1^{st} order FV flux on non-admissible subcells.

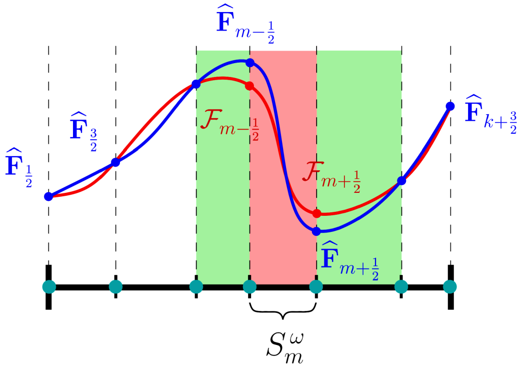


Figure: *A posteriori* correction on subcell $S_m^\omega \subset \omega$.

A priori approach

Goal. Combining reconstructed fluxes with 1^{st} order FV flux, i.e. introducing the following blended fluxes :

$$\widetilde{F}^{\omega_i}_{m\pm\frac{1}{2}} = F^{*,FV}_{m\pm\frac{1}{2}} + \theta_{m\pm\frac{1}{2}} \left(\widehat{F}^{\omega_i}_{m\pm\frac{1}{2}} - F^{*,FV}_{m\pm\frac{1}{2}} \right),$$

with $\theta_{m\pm\frac{1}{2}} \in [0,1]$ the coefficient assuring at local scale any convex property we want (entropy, preservation of maximum principle, water height positivity for Shallow-Water ...)
→ Theoretical proofs for preserving maximum principle.

Benefits.

- Simpler implementation opening the method to more people;
- No need to modify the neighbors of non-admissible cells, unlike *a posteriori* approach.

IV. Stabilization of Shallow-Water equations

Preservation of water-height positivity

For $m \in [\![1, k+1]\!]$ the following NSW dG formulation :

$$\int_\omega \partial_t \boldsymbol{v}_\omega \phi_m^\omega = -\int_\omega \partial_x \boldsymbol{F}_\omega \phi_m^\omega + \int_\omega \boldsymbol{B}_\omega \phi_m^\omega + \left[\phi_m^\omega (\boldsymbol{F}_\omega - \mathscr{F})\right]_{\partial\omega},$$

can be written as the *FV like* scheme on subcells :

$$\partial_t \overline{\boldsymbol{v}}_\omega = -\frac{1}{|S_m^\omega|}\left(\widetilde{\boldsymbol{F}}_{m+\frac{1}{2}}^\omega - \widetilde{\boldsymbol{F}}_{m-\frac{1}{2}}^\omega\right) + \overline{\boldsymbol{B}}_m^\omega,$$

with $\widetilde{\boldsymbol{F}}_{m+\frac{1}{2}}^\omega := \boldsymbol{F}_{m+\frac{1}{2}}^{*,FV} + \Theta_{m+\frac{1}{2}}\left(\widehat{\boldsymbol{F}}_{m+\frac{1}{2}}^\omega - \boldsymbol{F}_{m+\frac{1}{2}}^{*,FV}\right)$.

⚠ In order to use our stabilization method on Shallow-Water equations, we need to ensure that $\Theta_{m+\frac{1}{2}} = \mathrm{diag}(\theta_{m+\frac{1}{2}}^\eta, \theta_{m+\frac{1}{2}}^q)$ assure the preservation of water-height positivity and maximum principle → Theoretical proofs on intermediate Riemann states.

Internship continuation and Ph.D. goals

Internship continuation. Implementing the *a priori* stabilization method in my homemade `C++` code for scalar conservation law and in WaveBox for Shallow-Water equations.

Long term goals. Pursuing in Ph.D. to :

- Construct theoretical model for coupling those equations with a floating object in two dimension;
- Develop and implement the *a priori* correction for this problem and handle the coupling using an Arbitrary Lagrangian Eulerian (ALE) description.

Applications. Renewable energy, notably wave energy converters modeling and optimization.

Figure: *The Great Wave of Kanagawa*, Hokusai, 1830.

Thank you for your attention !